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Surface-wave generation by gusty wind

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Nikolayeva & Tsimring's (1986) collisionless Boltzmann model for surface-wave generation by a slowly fluctuating wind $U(z, t)$ is transformed to an equivalent steady flow in which the wind speed in the reference frame of the wave (of speed c) is given by $V(z) = \langle (U-c)^{-2}\rangle^{-1/2}$, where $\langle\ \rangle$ signifies a Gaussian average. This leads to a Sturm–Liouville equation for the Gaussian-averaged, complex amplitude of the wave-induced pressure. The wind-to-wave energy transfer for a logarithmic wind profile with the mean friction velocity $\kappa\bar{U}_1$ ($\kappa =$ Kármán's constant), the standard deviation $\delta\bar{U}_1$, and the roughness length $z_0 = \Omega\bar{U}_1^2/g$ is determined as a function of the parameters δ and Ω (Charnock's constant) through numerical integration of a Riccati equation (derived from the Sturm–Liouville equation). The energy transfer exceeds that predicted by the quasi-laminar model (Miles 1957; Conte & Miles 1959) by as much as 20–30% for $\delta \approx 1$ and c (wave speed) $\lesssim 6\bar{U}_1$ but is decreased for $c \gtrsim 8\bar{U}_1$ and may be negative for sufficiently large $c/\bar{U}_1$. These predictions contrast with the order-of-magnitude increase predicted by Nikolayeva & Tsimring.

1. Introduction

The quasi-laminar model for the transfer of energy from wind to surface waves (Miles 1957, hereinafter referred to as M57) is based on the Reynolds-averaged Euler equations and the implicit assumption that the characteristic period of the waves is long compared with that of the turbulent fluctuations. In fact, the time constants of wind gustiness may be large compared with those of wind-generated gravity waves, and the averaging of the Euler equations therefore should allow for a slowly and randomly varying wind. Different models for this accommodation have been proposed by Janssen (1986) and Nikolayeva & Tsimring (1986, hereinafter referred to as NT).

Janssen assumes the friction velocity U_* for a logarithmic wind profile to be random with the temporal mean $\bar{U}_*$ and the standard deviation σ_*. He then calculates the mean growth rate according to

$$\bar{\zeta}(\bar{U}_*, \sigma_*) = \int_{-\infty}^{\infty} G(U_*; \bar{U}_*, \sigma_*)\, \zeta(U_*)\, \mathrm{d}U_* \equiv \langle \zeta(U_*)\rangle, \tag{1.1}$$

where

$$G(U; \bar{U}, \sigma) = (2\pi)^{-1/2}\sigma^{-1} \exp\left[-\tfrac{1}{2}(U-\bar{U})^2/\sigma^2\right] \tag{1.2}$$

is a Gaussian distribution function, $\zeta(U_*)$ is the growth rate for a steady wind of friction velocity U_*, and $\langle\ \rangle$ signifies a Gaussian average. (Note that $\langle U\rangle \equiv \bar{U}$.) The integral in (1.1) may be approximated through Gauss–Hermite quadrature (Miles 1997).

Nikolayeva & Tsimring construct a more sophisticated model by analogy with the collisionless Boltzmann equation for the one-point distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ of the fluctuating velocity $\boldsymbol{v}$ at the space–time point $(\boldsymbol{r}, t)$ (Lundgren 1967). This leads them to a counterpart of the Rayleigh equation that governs the quasi-laminar model. However, whereas Rayleigh's equation is singular at $U = c$ (the *critical layer*, in which the wind-to-wave energy transfer is concentrated), this singularity is smoothed out in NT's model by the fluctuation of U. They conclude, from numerical integration of their equation, that gustiness may increase the wind-to-wave energy transfer, relative to that predicted by the quasi-laminar model, by one or two orders of magnitude. This is a matter of some practical importance in that an empirically modified version of the quasi-laminar model has been adopted as a basic component of the WAM model (Komen *et al.* 1994) for wind-wave prediction.

Against this background, and following M57 and NT, we consider here an incompressible, inviscid flow $U(z,t) - c$ over the gravity wave

$$z = a\cos kx \quad (ka \ll 1), \tag{1.3}$$

where x and z are horizontal and vertical Cartesian coordinates in a reference frame moving in the x-direction with the wave speed

$$c = (g/k)^{1/2}. \tag{1.4}$$

$U = U(z,t)$ is a slowly fluctuating wind speed with the temporal mean $\bar{U}(z)$, the standard deviation $\sigma(z)$ and the Gaussian distribution (1.2) and (by assumption) satisfies

$$|\partial U/\partial t| \ll kc|U|, \quad U(0,t) = 0. \tag{1.5a,b}$$

The basic formulation in §§2–4 is valid for any prescribed U that satisfies this description; however, (following M57 and NT) we assume a logarithmic profile for which

$$\bar{U}(z) = \bar{U}_1 \log(z/z_0) \quad (z \gg z_0), \quad \sigma/\bar{U}_1 = \text{constant} \equiv \delta, \quad \bar{U}(z_c) = c, \tag{1.6a–c}$$

$$\mathscr{z}_0 \equiv kz_0 \ll kz_c \equiv \mathscr{z}_c = \Omega e^{\mathscr{c}}/\mathscr{c}^2, \quad \mathscr{c} \equiv c/\bar{U}_1, \quad \Omega \equiv gz_0/\bar{U}_1^2, \tag{1.7a–c}$$

where Ω is Charnock's (1955) constant.

In §2, we consider two-dimensional perturbations about the flow $U(z,t)$† on the assumption of slowly and randomly modulated sinusoidal wave motion, transform NT's collisionless Boltzmann model to an equivalent steady flow with the equivalent wind speed (in the reference frame of the wave)

$$V(z) \equiv \langle (U-c)^{-2} \rangle^{-1/2}, \tag{1.8}$$

where $\langle\ \rangle$ is a Gaussian mean defined as in (1.1), and construct a Sturm–Liouville equation for the Gaussian mean of the complex amplitude of the perturbation pressure. In §3, we express V in terms of Dawson's integral.

In §4, we determine the inertial and energy-transfer parameters α and β defined by (as in M57)

$$P_0 \equiv (\alpha + i\beta)\, ka\bar{U}_1^2, \tag{1.9}$$

where P_0 is the Gaussian mean of the complex amplitude of the kinematic perturbation

† The assumption, implicit in §2, that the fluctuation of the wind is parallel to the mean wind is unrealistic. A model that allows for x- and y-components of the fluctuating wind and oblique propagation of the surface wave yields a description that differs from that of §2 only in the replacement of $\bar{U}$ by $\bar{U}\cos\theta$, where θ is the angle between the wave number $\boldsymbol{k}$ and the mean wind $\bar{\boldsymbol{U}}$.

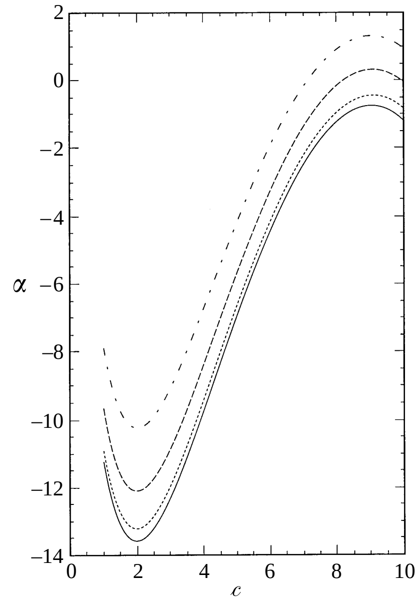

FIGURE 1. The inertial parameter α, as determined by the numerical integration of the Riccati equation (4.2) for the mean wind profile (1.6) with $\Omega = 0.003$ and $\delta = 0^+$ (——), 0.4 (· · · ·), 0.8 (– – –), 1.2 (–·–).

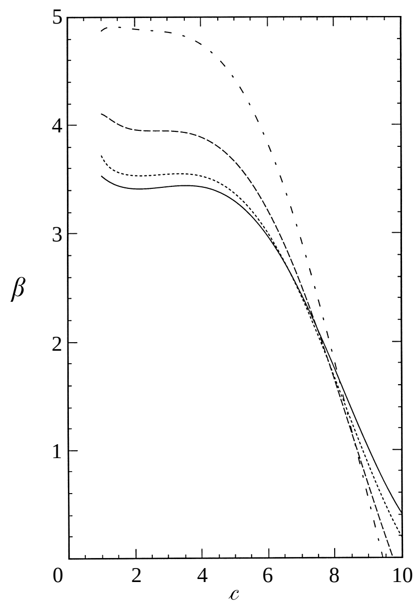

FIGURE 2. The energy-transfer parameter β, as determined by the numerical integration of the Riccati equation (4.2) for the mean wind profile (1.6) with $\Omega = 0.003$ and $\delta = 0^+$ (——), 0.4 (· · · ·), 0.8 (– – –), 1.2 (–·–).

pressure at the interface, through the numerical integration of a Riccati equation derived from the Sturm–Liouville equation for P; see figures 1 and 2. The dimensionless, wind-to-wave-energy-transfer rate is given by

$$(kc\bar{E})^{-1}(\overline{\partial E/\partial t}) = (\rho_a/\rho_w)(\bar{U}_1/c)^2\beta, \tag{1.10}$$

where $\bar{E}$ is the mean energy of the wave and ρ_a/ρ_w is the air/water density ratio. We find that gustiness may increase β for moderate c/U_1 but (in contrast to NT's prediction) decreases it, and may render it negative, for sufficiently large c/U_1.

2. Perturbation equations

2.1. *NT's model*

We pose the velocity and the perturbation pressure in the forms

$$\boldsymbol{v} = [U(z,t) - c, 0] + \mathrm{Re}\{[u_1(z,t), w_1(z,t)]\,\mathrm{e}^{\mathrm{i}kx}\}, \tag{2.1a}$$

$$p = \mathrm{Re}\{p_1(z,t)\,\mathrm{e}^{\mathrm{i}kx}\}, \tag{2.1b}$$

where p_1, u_1 and w_1 are slowly fluctuating, complex amplitudes. NT's collisionless Boltzmann model yields the perturbation equations [their (8)]

$$\mathrm{i}k\langle w_1\rangle = -A\rho_a^{-1}D\langle p_1\rangle \quad (D \equiv \mathrm{d}/\mathrm{d}z), \tag{2.2a}$$

$$\mathrm{i}k\langle u_1\rangle + (A^{-1}DA - B^{-1}DB)\langle w_1\rangle = -\mathrm{i}kA\rho_a^{-1}\langle p_1\rangle, \tag{2.2b}$$

$$\mathrm{i}k\langle u_1\rangle + D\langle w_1\rangle = 0, \tag{2.2c}$$

after the replacement of their carrier $\exp(-\mathrm{i}kx)$ by the present $\exp(\mathrm{i}kx)$, their Gaussian means $\bar{p}_{a1}$, $\bar{u}_{a1}$, $\bar{w}_{a1}$ by $\langle p_1\rangle$, $\langle u_1\rangle$, $\langle w_1\rangle$, and their $-ka$ and $-kb$ by

$$A \equiv \langle (U-c)^{-1}\rangle, \quad B \equiv \langle (U-c)^{-2}\rangle = \partial A/\partial c. \tag{2.3a, b}$$

The elimination of $\langle p_1\rangle$ and $\langle u_1\rangle$ from (2.2) yields [NT's (9)]

$$AD[B^{-1}D(A^{-1}B\langle w_1\rangle)] - k^2\langle w_1\rangle = 0. \tag{2.4}$$

2.2. *Equivalent steady flow*

Let

$$\langle p_1\rangle = \rho_a P, \quad \langle u_1\rangle = AV\hat{U} - (\mathrm{i}k)^{-1}(DAV)\,W, \; \langle w_1\rangle = AVW, \tag{2.5a–c}$$

where $V = B^{-1/2}$, as anticipated in (1.8), and the hat distinguishes the dependent variable $\hat{U}$ from the basic wind speed U. Transforming (2.2), we obtain

$$\mathrm{i}kVW = -DP, \quad \mathrm{i}kV\hat{U} + (DV)\,W = -\mathrm{i}kP, \quad \mathrm{i}k\hat{U} + DW = 0, \tag{2.6a–c}$$

which describe an equivalent steady flow in which $\hat{U}$, W and P are the complex amplitudes of the perturbation velocity and kinematic pressure. Eliminating $\hat{U}$ and either P or W from (2.6), we obtain either the Rayleigh equation [cf. (2.4)]

$$D^2W - (k^2 + V^{-1}D^2V)\,W = 0, \tag{2.7}$$

for which the interfacial (tangential flow) and null conditions are

$$W = -\mathrm{i}kac \quad (z = 0), \quad W \to 0 \quad (kz \uparrow \infty), \tag{2.8a, b}$$

or the Sturm–Liouville equation

$$D(BDP) - k^2BP = 0, \tag{2.9}$$

for which the boundary conditions are

$$BDP = k^2a \quad (z = 0), \quad P \to 0 \quad (kz \uparrow \infty). \tag{2.10a, b}$$

3. Analytical representation of A and B

Returning to (2.3), recalling G from (1.2), introducing

$$\xi = (U-\bar{U})/\surd 2\sigma, \quad \nu = (\bar{U}-c)/\surd 2\sigma, \tag{3.1 a, b}$$

and assuming that σ is constant (which requires $z \gg z_0$), we obtain

$$A(\nu) \equiv \langle (U-c)^{-1} \rangle = \int_{-\infty}^{\infty} G(U; \bar{U}, \sigma)\,(U-c)^{-1}\,\mathrm{d}U \tag{3.2 a}$$

$$= (2\pi)^{-1/2}\sigma^{-1}\int_{-\infty}^{\infty} (\xi+\nu)^{-1}\,\mathrm{e}^{-\xi^2}\,\mathrm{d}\xi \tag{3.2 b}$$

$$= 2^{1/2}\sigma^{-1}[D(\nu)+\tfrac{1}{2}\mathrm{i}\pi^{1/2}\,\mathrm{e}^{-\nu^2}], \tag{3.2 c}$$

$$B(\nu) = \partial A/\partial c = \sigma^{-2}[-(\mathrm{d}D/\mathrm{d}\nu)+\mathrm{i}\pi^{1/2}\nu\,\mathrm{e}^{-\nu^2}], \tag{3.3}$$

where $D(\nu)$ is Dawson's integral (Spanier & Oldham 1987, §42).

4. Riccati formulation

We find it expedient for numerical integration to introduce

$$R(\varkappa) \equiv kQ(DP)^{-1}P, \quad \varkappa \equiv kz, \quad Q \equiv (\bar{U}_1^2 B)^{-1} \tag{4.1 a–c}$$

and transform (2.9) to the Riccati equation

$$\mathrm{d}R/\mathrm{d}\varkappa = Q - Q^{-1}R^2. \tag{4.2}$$

Invoking (1.9), (2.10 a) and (4.1 a) and projecting the inner boundary condition on $\varkappa = \varkappa_0$ to satisfy $\bar{U} = 0$, we obtain

$$\alpha + \mathrm{i}\beta = P_0/ka\bar{U}_1^2 = R_0 \quad (\varkappa = \varkappa_0). \tag{4.3}$$

The null condition (2.10 b) requires $DP \sim -kP$, or, through (4.1 a),

$$R \sim -Q \quad (\varkappa \uparrow \infty). \tag{4.4}$$

The numerical integration of (4.2), subject to (4.4), may be started from the asymptotic expansion

$$R = \sum_{n=0}^{\infty} \varkappa^{-n} F_n(\lambda), \quad \lambda = (\bar{U}-c)/\bar{U}_1 = \ln(\varkappa/\varkappa_c), \tag{4.5 a, b}$$

where

$$F_0 = -Q, \quad F_1 = \tfrac{1}{2}(\mathrm{d}F_0/\mathrm{d}\lambda), \tag{4.6 a, b}$$

$$2F_{n+1} = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} - n\right)F_n + Q^{-1}\sum_{m=1}^{n} F_m F_{n+1-m} \quad (n = 1, 2, \ldots), \tag{4.6 c}$$

for some sufficiently large value of $\varkappa$ and continued inward to $\bar{U} = 0$ at $\varkappa_0 = \Omega/c^2$ (where $\Omega =$ Charnock's constant and $c \equiv c/\bar{U}_1$) to determine R_0. Alternatively, the integration with respect to $\varkappa$ may be stopped at $\varkappa = \varkappa_c = \Omega \mathrm{e}^{c}/c^2$ and

$$\mathrm{d}R/\mathrm{d}\lambda = \varkappa_c\,\mathrm{e}^{\lambda}(Q - Q^{-1}R^2) \tag{4.7}$$

integrated from $\lambda = 0$ ($\varkappa = \varkappa_c$) to $\lambda = -c$ ($\varkappa = \varkappa_0$). The truncation of (4.5 a) at $n = 2$ yields

$$R \sim -Q\left\{1 + \tfrac{1}{2}(\varkappa Q)^{-1}\frac{\mathrm{d}Q}{\mathrm{d}\lambda} + \tfrac{1}{4}(\varkappa Q)^{-2}\left[Q\left(\frac{\mathrm{d}^2Q}{\mathrm{d}\lambda^2} - \frac{\mathrm{d}Q}{\mathrm{d}\lambda}\right) - \tfrac{1}{2}\left(\frac{\mathrm{d}Q}{\mathrm{d}\lambda}\right)^2\right]\right\}. \tag{4.8}$$

5. Numerical solution

A fast and reliable pair of Fortran routines for computing the Dawson integral has been written by Wayne Fullerton for the Slatec library. The double precision version is *daws.f*, the extended precision (32 digit) one is *ddaws.f*. (We used the conventional double precision version for our computations.) These and the several required subsidiary subroutines are readily available at the Netlib software repository on the Internet, located at *http://www.netlib.no/*. Derivatives of the Dawson integral were evaluated through the recurrence relation

$$\frac{\mathrm{d}D(\nu)}{\mathrm{d}\nu} = 1 - 2\nu D(\nu), \tag{5.1}$$

obviating the need for any explicit numerical differentiation.

A check on the numerical solution of (4.2) was effected by regarding it as an exact result from which to determine the error of the three- and six-term truncations of (4.5 *a*) as a function of z. As one would expect, the ratio of the inferred error curves produces essentially a straight line of slope -3 when plotted as a function of z on log–log axes. (While these series solutions can, for large c, be continued somewhat beyond z_c in the direction of z_0, the determination of α and β apparently requires matching with an expansion pivoted about $z = z_c$). We found that $z = 15$ was a satisfactory point at which to commence integration, with larger values producing essentially no change in the determination of α and β.

An independent check on the numerical results was obtained from the limiting behaviour of a sequence of (α, β) pairs for δ tending to zero. The limiting values were extrapolated from a low-order, least-square-polynomial fit near $\delta = 0$. In all cases tested, values were in agreement with the entries given in Conte & Miles (1959) to the tabulated accuracy. (Note that the curves labelled 0^+ in figures 1 and 2 are actually computed at $\delta = 0.01$. On the scale plotted, adjustment by extrapolation to the inviscid result would make an imperceptible graphical change.)

For solution of the Ricatti equation (4.2), we used an adaptive step-size integrator based on repeated Richardson extrapolation (Dahlquist & Bjorck 1974). There is no difficulty in achieving a local absolute error of 10^{-12}; the routine automatically accommodates the change in character of the solution across the critical layer.

The parameters α and β for $\Omega = 0.003$ (NT's value), are plotted in figures 1 and 2 for $\delta = 0$, 0.4, 0.8 and 1.2. The results for $\delta = 0$ agree with those of Conte & Miles (1959). NT's estimate of $\sigma = 2.3\bar{U}_* = 0.94\bar{\mathrm{U}}_1$ (for $\kappa = 0.4$) implies $\delta = 0.94$, which is bracketed by the present results for $\delta = 0.8$ and 1.2.

6. Conclusion

The ratio $\beta(\delta)/\beta(0)$ implied by the present figure 2 is significantly smaller than the ratio implied by NT's figures 1 and 2. In particular, for $c \gtrsim 0.9$ and $\delta = 1.2$, the present ratio is < 1, in contrast to NT's 80. Moreover, in contrast to the prediction of the quasi-laminar model that β is positive-definite for $\delta = 0$, the present model predicts that $\beta(\delta)$ may be negative for sufficiently large c (see Appendix).

We conclude that the wind-to-wave energy transfer predicted by NT's model may exceed that predicted by the quasi-laminar model by as much as 20–30 % for $c \lesssim 6$, but may be inferior thereto for $c \gtrsim 8$. We do not understand the order-of-magnitude increase predicted by NT but suggest that their numerical evaluation of the singular integrals A and B (their $-ka$ and $-kb$) may be less reliable than our analytical reduction in (3.2) and (3.3) above. Finally, we remark that our predicted gust-induced increase (for moderate c/U_1) in the energy transfer relative to that predicted by the quasi-laminar model is commensurate with observation (Komen *et al.* 1994, p. 72), whereas there is no (currently accepted) observational support for an order-of-magnitude increase.

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Appendix. Quadratic integral for β

Multiplying (2.9) through by P^*, the complex conjugate of P, integrating by parts over $0 < z < \infty$, and invoking (2.10 *a, c*), we obtain

$$k^2 a P_0^* = -\int_0^\infty B(|DP|^2 + k^2|P|^2)\,\mathrm{d}z. \tag{A 1}$$

The imaginary part of (A 1) yields

$$\beta \equiv \mathrm{Im}\left(\frac{P_0}{ka\bar{U}_1^2}\right) = \int_0^\infty \frac{B_i(|DP|^2 + k^2|P|^2)\,\mathrm{d}z}{k^3 a^2 \bar{U}_1^2} \tag{A 2a}$$

$$= \pi^{1/2}(\bar{U}_1/\sigma)^2 \int_0^\infty \nu\,\mathrm{e}^{\nu^2} \Pi(\varkappa)\,\mathrm{d}\varkappa \;(\varkappa \equiv kz), \tag{A 2b}$$

where (A 2*b*) follows from (A 2*a*) through (3.3), $\nu = (\bar{U} - c)/\surd 2\sigma$, and

$$\Pi(\varkappa) \equiv \frac{|DP|^2 + k^2|P|^2}{(k^2 a \bar{U}_1^2)^2}. \tag{A 3}$$

$\Pi(\varkappa)$ is positive-definite, and β is positive if the contributions of the integrand for $\varkappa > \varkappa_c$ ($\nu > 0$) outweigh those for $\varkappa < \varkappa_c$ ($\nu < 0$), as proves to be the case for $\varkappa_c \ll 1$. However, Π decays like $\exp(-2\varkappa)$ for $\varkappa > \varkappa_c \gtrsim 1$, in consequence of which β may be negative if $\delta > 0$. (The limit $\delta \downarrow 0$ is singular.)

If $\delta \ll 1$, the integrand of (A 2*b*) is concentrated near $\varkappa = \varkappa_c$ ($\nu = 0$), and integration by parts, together with $\mathrm{d}\nu/\mathrm{d}\varkappa = 1/\surd 2\delta\varkappa$ and $\delta \equiv \sigma/\bar{U}_1$, yields (after a non-trivial reduction)

$$\beta = \pi^{1/2} \int_{\nu_0}^\infty \mathrm{e}^{-\nu^2} \varkappa(\varkappa\Pi)_\varkappa\,\mathrm{d}\nu \tag{A 4a}$$

$$\sim \pi(kz_c)^{-1}|W(z_c)/ka\bar{U}_1|^2 \quad (\delta \downarrow 0) \tag{A 4b}$$

in agreement with the quasi-laminar result (M57).

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